

Jump Locations of Jump-Diffusion Processes with State-Dependent Rates

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We propose a general framework for studying jump-diffusion systems driven by both Gaussian noise and a jump process with state-dependent intensity. Of particular natural interest are the jump locations: the system evaluated at the jump times. However, the state-dependence of the jump rate provides direct coupling between the diffusion and jump components, making disentangling the two to study individually difficult. We provide an iterative map formulation of the sequence of distributions of jump locations. Computation of these distributions allows for the extraction of the interjump time statistics. These quantities reveal a relationship between the long-time distribution of jump location and the stationary density of the full process. We provide a few examples to demonstrate the analytical and numerical tools stemming from the results proposed in the paper, including an application that shows a non-monotonic dependence on the strength of diffusion.

I. INTRODUCTION

Stochastic models driven by both Gaussian noise (diffusion) and a discrete jump component, or so called *jump-diffusion* models have been used in a wide variety of applications. Perhaps most commonly, these jump-diffusion models are used in mathematical finance [2–5] to describe frequent small transactions with occasional larger ones. Jump-diffusion descriptions have also served useful in neuroscience, typically describing the noisy buildup of voltage in a neuron leading to stochastic firing [6–9]. Other applications include biophysical descriptions of movement of chromosomes [10] and the interaction between soil moisture and rainfall events [11]. Also relevant to the results described in this paper is a subset of jump-diffusion models that neglect Gaussian noise and are driven by the dynamics of the jump process alone. This type of models appears in an equally eclectic variety of applications [12–15].

Of particular interest in this work are jump-diffusion processes with state-dependent jump rates, which appear in some previous work [13, 14, 16]. This is a natural generalization, as any realistic system is unlikely to have a constant jump rate, but rather one that depends on the current state of the system. For instance, a neuron is more likely to fire as the voltage increases or, in finance, an asset to crash as it rises in price.

Previous work has thoroughly investigated jump-diffusion models for constant jump rates [1, 11]. The interjump time distributions for discrete noise alone has also been studied [13, 16] primarily using renewal theory. However, in general, state-dependent jump-diffusion is *not* a renewal process, as sequential distributions are not independent. The theory of Cox processes [17] is also well-established and refers to a diffusion process driving a state-dependent Poisson process, but this does not complete the feedback loop of the jump process further

modifying the state, which jump-diffusion allows. Diffusion with switching, which bares resemblance to jump-diffusion, has also been studied [18] but is distinctly different in the role of discrete noise and not addressed in this work.

It is worth mentioning the issue of blowup in these models. That is, for some choices of the diffusion and jump components, the system will experience an infinite number of jumps or blowup (or both) in finite time, or is said to be unstable. Although thoroughly studied [19–21] and shown subtle even for relatively simple models, stability is deliberately not focused upon in this work. However, in the appendix of this work, we present some thoughts on how the proposed framework could provide a unique perspective in studying these phenomena, as this is primarily a partial differential equations (PDEs) formulation rather than the existing path-wise approaches.

If a jump process is state-dependent, then the distribution (or statistics) of the jump *locations* can be studied. That is, we refer to the locations as the state of the system evaluated at the jump times. For instance, this could describe the statistics of the voltage at which a neuron fires or the price at which a jump occurs in finance. This seemingly natural quantity of interest is seemingly unstudied and the main focus of this work. Although efficient Monte Carlo methods exist for jump-diffusion and Lévy processes [22, 23], the jump component may be rare and therefore difficult to find an accurate distribution of. A full PDE description of the process can also be computed, but it is seemingly difficult (or impossible) to disentangle the jump component alone.

In this work, we present a general (accommodating a wide variety of models) framework for studying jump-diffusion systems and particularly focus on studying the sequence of jump locations. We present an iterative map formulation explicitly describing the distribution of the i th jump location, which allows for statistics to be extracted about the interjump times even with diffusion, a limitation of previous works [13]. By assuming the process reaches stationarity, we find identify term behavior of the distribution of jump locations and statistics

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of the interjump times. The stationary jump locations turn out to be closely related to the full stationary process, being equal if and only if the system is trivially state dependent. A few examples are provided, illustrating the usefulness of the theory toward analytical and numerical results. One example provides an interesting non-monotonic effect of diffusion on the interjump time statistics in a particular model.

II. FORMULATION

A. Jump-Diffusion

Let X_t denote a state-dependent jump-diffusion process and $\tilde{p}(x, t)$ be the probability density (PDF) of this process. The evolution of the density is described by the forward Chapman-Kolmogorov equation [24, 25]

$$\partial_t \tilde{p}(x, t) = \mathcal{L}\tilde{p} - \lambda(x)\tilde{p} + \mathcal{J}\lambda(x)\tilde{p}, \quad (1)$$

where the operator \mathcal{L} characterizes the diffusion component, $\lambda(x)$ is the state-dependent intensity at which the jump process occurs, and the operator \mathcal{J} describes the jump. A feature we seek to emphasize is the state dependence of the jump-rate $\lambda(x)$, resulting in direct coupling between the diffusion and jump components of the process.

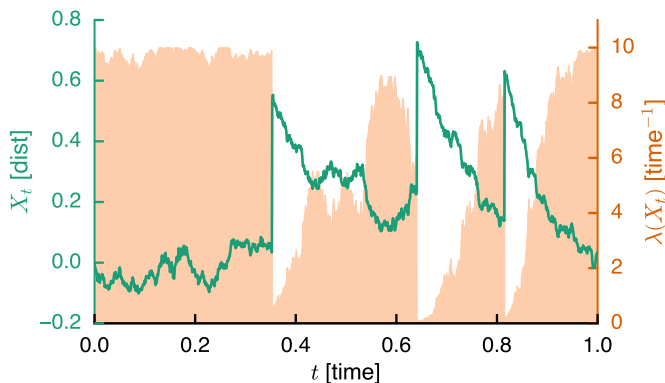


FIG. 1. An example realization of a jump-diffusion process X_t with $\mathcal{L}p = \partial_x \{axp\} + Dp_{xx}$, $\lambda(x) = \alpha \exp \{-x^2/\beta\}$ and $\mathcal{J}p = p(x - \Delta)$. Also shown is the state dependent jump rate $\lambda(x)$. In this particular realization, 3 jumps occur. Parameter values used $a = .1, D = 1, \alpha = 10, \beta = 10, \Delta = 1$.

Although the particular change choices of \mathcal{L} and \mathcal{J} significantly change the behavior and characteristics of the process, the description (1) is flexible enough to accommodate a wide variety of models, which we will now elaborate upon.

B. Diffusion Component

Consider the classical one-dimensional SDE driven by Brownian motion described by

$$dY_t = A(Y_t) + \sqrt{2D(Y_t)} dW_t, \quad (2)$$

where W_t is a Brownian motion. The Fokker-Planck (forward Chapman-Kolmogorov) equation describing the evolution of the probability density function $p(y, t)$ corresponding to the path-wise description (2) is

$$\partial_t p(y, t) = \mathcal{L}p := -\partial_y \{A(y)p\} + D(y)\partial_{yy}p.$$

We use the *Fokker-Planck operator* \mathcal{L} to characterize the diffusion process for the remainder of the paper.

C. Jump Component

Let Z_t be an inhomogeneous (state-dependent) Poisson process with intensity (rate) $\lambda(X_t)$. Also necessary is a description of what actually occurs at the jumps, sometimes referred to as the *reset map* [20]. The behavior of the reset map is characterized by the *jump operator* \mathcal{J} , which is a probability density flux, ensuring that (1) indeed describes the evolution of a probability density. We now describe a few possible choices of this operator.

1. Jump Operator Examples

1. *Constant jump size.* At the jump times, the process increments by a fixed amount Δ , so

$$X \rightarrow X + \Delta.$$

Then, the corresponding jump operator \mathcal{J} is then a shift by that fixed quantity

$$\mathcal{J}p := p(x - \Delta).$$

2. *Reset.* Fundamentally different than the previous example, rather than jumping by a fixed displacement, the process resets to a deterministic quantity, η , so

$$X \rightarrow \eta.$$

The corresponding \mathcal{J} operator is

$$\mathcal{J}p := \delta(x - \eta) \int_{-\infty}^{\infty} p(x, t) dx,$$

where $\delta(x)$ is the Dirac delta function, again noting that the scaling ensures flux preservation.

3. *Random jump size.* A generalization of the first example, the particle can also jump a random displacement, where the size of the jump is described

by the probability density $\mu(\Delta, x)$, which can also be state-dependent. The \mathcal{J} operator is

$$\mathcal{J}p := \int_{-\infty}^{\infty} p(x - \Delta, t) \mu(\Delta, x) d\Delta.$$

4. *Other examples.* Although not directly considered in this paper, reset maps of the form

$$X \rightarrow \gamma X \quad \implies \quad \mathbb{J}p := p(\gamma x)$$

have been studied in other work [20] and can be formulated in our framework in the described manner. A generalization of the reset case could also be made to allow for resets to a random location, but this does not seem to provide additionally interesting structure.

It is worth noting that in the reset case, the system is indeed a renewal process, as the distributions between jumps are entirely independent, meaning that the tools of [11, 13, 16] can be utilized. This reset case has also been studied more explicitly in [26, 27].

III. RESULTS

A. Jump Distributions

A more convenient process to study is what we refer to as the *absorbing process*, described by

$$\begin{cases} \partial_t p(x, t) = \mathcal{L}p - \lambda p \\ \partial_t q(x, t) = \lambda p. \end{cases} \quad (3)$$

Note that we distinguish between this and the full process $\tilde{p}(x, t)$, which we refer to henceforth as the *reinjected process*. While the absorbing process does not capture all of the behavior of the full reinjected process, the absorbing process fixes the distribution (in both space and time) of jump events and consequently serves more fruitful in disentangling the jump component from the full process.

Theorem 1. *The densities of the first jump location $p_j(x)$ and first jump time $p_\tau(t)$ of (3) are described by*

$$p_j(x) = q(x, \infty) = \int_0^\infty \lambda(x) p(x, t) dt$$

and

$$p_\tau(t) = \int_{-\infty}^{\infty} \lambda(x) p(x, t) dx.$$

Proof. Define the survival probability $s(t)$ be the probability that the first jump *has not* occurred by time τ

$$s(t) := \mathbb{P}[\tau > t] = \int_{-\infty}^{\infty} p(x, t) dx,$$

which means the first jump time density $p_\tau(t)$ is then

$$p_\tau(t) = -\frac{ds}{dt} = -\int_{-\infty}^{\infty} \partial_t p dx.$$

Then, using (3) and the fact that $p \rightarrow 0$ as $x \rightarrow \pm\infty$, we obtain

$$p_\tau(t) = \int_{-\infty}^{\infty} \lambda p dx = \int_{-\infty}^{\infty} \partial_t q dx.$$

Due to the lack of spatial flux in $q(x, t)$, the density of exit locations is $q(x, \infty)$, which can be obtained by integrating over all possible jump times

$$p_j(x) = \int_0^\infty \partial_t q dt = q(x, \infty) - q(x, 0) = q(x, \infty)$$

by noting that $q(x, 0) \equiv 0$. \square

Although this provides useful information about the distributions (in time and space) at which a particular (the first) jump occurs, we are interested in studying the distributions of all jumps.

B. Jump Location Sequential Mapping

We instead study the interjump dynamics by noting that between jumps, the full process (1) can be described by the absorbing process (3). Define t_i to be the i th jump time and let $X_i := X_{t_i}$ be the i th jump location. Consider the iterative definition of $p_i(x)$ for $i = 2, 3, \dots$

$$\begin{cases} \partial_t \hat{p}_i(x, t) = \mathcal{L}\hat{p}_i - \lambda \hat{p}_i \\ \hat{p}_i(x, 0) = \mathcal{J}p_{i-1} \\ p_{i+1}(x) = \int_0^\infty \lambda \hat{p}_i dt. \end{cases} \quad (4)$$

There is a slight clumsiness with the first iteration of this procedure. Let p_0 be some known starting distribution of the process, then the first iterate becomes

$$\begin{cases} \partial_t \hat{p}_1(x, t) = \mathcal{L}\hat{p}_1 - \lambda \hat{p}_1 \\ \hat{p}_1(x, 0) = p_0 \\ p_1(x) = \int_0^\infty \lambda \hat{p}_1 dt, \end{cases}$$

noting the difference in the starting conditions.

Theorem 2. *The distribution of the i th jump location X_{t_i} is $p_i(x)$ and is described iteratively by (4).*

This construction follows from directly from **Theorem 1**. Effectively, \hat{p}_i serves as intermediate quantity tracking the distribution of all possible jump locations (and times) for that particular iterate. Then, to start the next iterate, the distribution of jump locations must be modified jump procedure characterized by \mathcal{J} . Although tracking the jump locations rather than the jump times seems counterintuitive at first, no natural analogous formulation is apparent.

Define a more convenient quantity to study

$$u_i := p_i(x)/\lambda(x). \quad (5)$$

If $\lambda(x) = 0$ for some x , then necessarily $p_i(x) = 0$ since $\lambda(x) = 0$ implies no jump can occur at this location, hence this quotient causes no difficulties.

Theorem 3. *The description (4) is equivalent to*

$$\mathcal{T}u_{i+1} := [\lambda - \mathcal{L}]u_{i+1} = \mathcal{J}\lambda u_i,$$

or more explicitly

$$u_{i+1} = \mathcal{T}^{-1}\mathcal{J}\lambda u_i. \quad (6)$$

Proof. Integrating both sides of (4) with respect to t and noting that $\hat{p}_i(x, \infty) = 0$ we are left with

$$\hat{p}_i(x, 0) = \int_0^\infty \mathcal{L}\hat{p}_i dt - \lambda(x) \int_0^\infty \hat{p}_i.$$

The linear operator \mathcal{L} is a differential operator in x and consequently commutes with the time integral. Using the initial condition and $p_i/\lambda = \int_0^\infty \hat{p}_i dt$, we obtain the desired result. \square

The map (6) provides an explicit description for the sequence

$$u_1(x) \rightarrow u_2(x) \rightarrow \cdots \rightarrow u_\star(x) \rightarrow u_\star(x) \rightarrow \cdots, \quad (7)$$

where, we are assuming the process reaches stationarity. Convergence of the sequence (7) corresponds to a fixed point of the map (6). Although we seek not to focus on this aspect, a natural question to ask is under what conditions the process reaches stationarity, or equivalently, when does u_i converge to some u_\star . In **Appendix B**, we provide a brief commentary about how the relationship (6) can be thought of as an iterated linear non-negative integral operator which can be studied as such. Results from theoretical ecology literature are then cited which provide a heuristic analysis of conditions for convergence.

Proceeding assuming the iterations converge to stationarity, we can assume that (6) has a fixed point, which must be of the following form.

Theorem 4. *The stationary distribution of the jump locations $p_\star(x)$ is described by*

$$0 = \mathcal{L}u_\star - \lambda u_\star + \mathcal{J}\lambda u_\star, \quad (8)$$

where $u_\star := p_\star/\lambda$.

Corollary 1. *The stationary distribution of the jump locations p_\star is the same as the stationary distribution of the full process if and only if λ is constant.*

Proof. This is an immediate consequence of taking the full process to be in stationarity, so $d\hat{p}/dt = 0$ in (1), which satisfies

$$0 = \mathcal{L}\hat{p}_s - \lambda\hat{p}_s + \mathcal{J}\lambda\hat{p}_s. \quad (9)$$

This is exactly the same relationship as (8). Recalling that $\lambda u_\star = p_\star$ and that both p_\star and \hat{p}_s are both probability densities, the only way that $p_\star = \hat{p}_s$ is if u_\star is a rescaling of p_\star or, equivalently, λ is constant. \square

Elaborating a bit on this relationship: while (9) and (8) may appear to produce the same result, the solutions to each require different scalings. Since \hat{p}_s is a probability density, it must be that

$$\int_{-\infty}^\infty \hat{p}_s dx = 1. \quad (10)$$

However, $u_\star\lambda = p_\star$, where p_\star is a probability density, so this means that

$$\int_{-\infty}^\infty u_\star(x)\lambda(x) dx = 1. \quad (11)$$

C. Moments

Define τ_i to be the i th interjump time. That is, $\tau_i := t_i - t_{i-1}$. Also, let τ_\star be the interjump time for the stationary process.

Theorem 5. *The mean interjump time τ_i can be recovered from the distribution of the i th jump location and is*

$$\langle \tau_i \rangle = \int_{-\infty}^\infty u_i dx = \int_{-\infty}^\infty \frac{p_i(x)}{\lambda(x)} dx. \quad (12)$$

Also, the mean stationary interjump time can be computed from the stationary distribution of the jump locations p_\star

$$\langle \tau_\star \rangle = \int_{-\infty}^\infty u_\star dx = \int_{-\infty}^\infty \frac{p_\star(x)}{\lambda(x)} dx. \quad (13)$$

Proof. We integrate both sides of (4) with respect to x , again noting that $\hat{p}_i \rightarrow 0$ as $x \rightarrow \pm\infty$, resulting in

$$\partial_t \int_{-\infty}^\infty \hat{p}_i dx = - \int_{-\infty}^\infty \lambda \hat{p}_i dx.$$

However, from **Theorem 1**, we see that the right-hand side is exactly the distribution of the interjump time p_{τ_i} , so we have

$$\partial_t \int_{-\infty}^\infty \hat{p}_i dx = -p_{\tau_i}(t) \quad (14)$$

Taking the mean on both sides with respect to τ_i ,

$$\langle \tau_i \rangle = - \int_0^\infty \int_{-\infty}^\infty t \partial_t \hat{p}_i dx dt,$$

after integrating by parts and noting that $\int_0^\infty \hat{p}_i dt = p_i/\lambda$, we get the desired result. \square

While the first moment (mean) of the interjump time can be computed directly with a quadrature of the presumed known p_i , the higher order moments are less straightforward. **Appendix A** describes how, in theory, knowledge of the p_i can be used to extract higher order

moments of τ_i . Solving for higher order moments using this approach requires solving a hierarchy of differential equations, meaning in practicality, this may not be so feasible. However, a more practical numerical approach may be to solve for p_i and then solve the absorbing (3) process run to extract explicitly p_{τ_i} using the relationships from **Theorem 1**.

IV. EXAMPLES

In this section, we briefly provide some examples to illustrate the usefulness of the proposed results. The first, a relatively simple case demonstrates the ability to derive analytical quantities for models which were difficult to study previously. The second example provides a brief analysis identifying that certain models may not be feasible to study with Fourier analysis directly but become so when formulated in our framework. The third and final example, which is studied numerically from the relationships proposed in this work, provides interesting interactions between the state-dependence of the jump process and the strength of diffusion.

A. Constant drift with reset to origin

The reset case has been studied in other works [26, 27], but will serve here to illustrate the ability to derive exact quantities utilizing the proposed framework. Consider a jump-diffusion model with constant drift and diffusion with jump component to be reset to zero with rate $\lambda(x) = \exp\{x\}$. Note that the reset structure means that the process is a renewal process, and consequently $p_i = p_*$ for all i . That is, the process is in stationarity after a single jump. Then, in stationarity, the fundamental relation in this work (8) becomes

$$0 = -\partial_x \{au_*\} + D\partial_x xu_* - e^x u_* + \delta(x) \int_{-\infty}^{\infty} e^x u_* dx. \quad (15)$$

From (11), the integral scaling the δ function is simply equal to 1, so

$$\delta(x) = -\partial_x \{au_*\} + D\partial_x xu_* - e^x u_*. \quad (16)$$

This can be computed in a similar manner to a Green's function, noting that for $x \neq 0$, we have

$$0 = -\partial_x \{au_*\} + D\partial_x xu_* - e^x u_*, \quad (17)$$

and integrating (16) from $(-\varepsilon, \varepsilon)$ and using the fact that u_* must be continuous, we get the matching condition

$$D \{u'(0^+) - u'(0^-)\} = -1. \quad (18)$$

By solving (17), we get that our solution is of the form

$$u_* = \begin{cases} c_1 u_L(x) := c_1 e^{\frac{ax}{2b}} I_{a/b} \left(\frac{2\sqrt{b}e^x}{b} \right) & x < 0 \\ c_2 u_R(x) := c_2 e^{\frac{ax}{2b}} K_{-a/b} \left(\frac{2\sqrt{b}e^x}{b} \right) & x > 0, \end{cases}$$

where I, K are Bessel functions [28] such that $I \rightarrow 0$ as $x \rightarrow -\infty$ and $K \rightarrow 0$ as $x \rightarrow \infty$ (and are linearly independent). The jump condition (18) then becomes

$$D \{c_2 u'_R(0) - c_1 u'_L(0)\} = -1. \quad (19)$$

However, we also need to impose continuity of u_* , so we also have the requirement

$$c_1 u_L(0) = c_2 u_R(0). \quad (20)$$

The conditions (19),(20) provide us two equations for two unknowns, which yield

$$c_1 = \frac{2}{D} 2K_{a/D}(2/\sqrt{D}), \quad c_2 = \frac{2}{D} 2I_{a/D}(2/\sqrt{D}).$$

Thus far, our solution is then defined only up to a constant, but we know that u_* must satisfy the scaling (11), which our choice of c_1, c_2 serendipitously already satisfy.

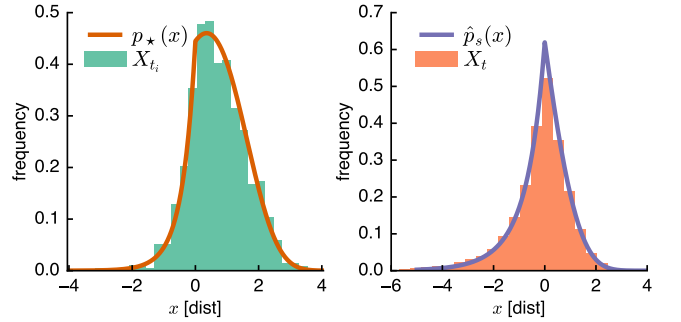


FIG. 2. A comparison of Monte Carlo simulations of the example described by (15) and analytically derived corresponding quantities. Left: The stationary jump location density $p_*(x)$ is compared with a histogram of realizations of the state of the process X_t evaluated at the jump times t_i . Right: The derived stationary distribution of the full process \hat{p}_s is compared with the histogram of a realization of the process X_t . Parameter values used are $a = 1, D = 1$.

By computing u_* explicitly, we can use (5) to immediately obtain the distribution of jump locations p_* . From (13), we could also, in principle, compute statistics about the interjump times. Finally, from **Corollary 1**, we also have the stationary density of the full process, \hat{p}_s , which is a rescaling of q_* such that (10) is satisfied. We seek to emphasize that being able to compute the distribution of stationary jump locations or the stationary density itself (either analytically as done in this example or numerically) allows for the extraction of the other, illustrating the usefulness and practicality of our results.

In Figure 2, a comparison is shown between Monte Carlo simulations of the process X_t corresponding to (15) and the analytically derived distributions in this section. Specifically, an Euler-Maruyama [29] scheme corresponding to the path-wise description was simulated, recording all of the values on the realization of the path as well as the values of X_t at the jump times t_i . The figure illustrates that both the analytically derived expressions

for the stationary jump locations $p_\star(x)$ and the stationary distribution of the full process $\hat{p}_s(x)$ agree with the Monte Carlo simulations.

B. Constant drift with random jump size

This example is presented only briefly to illustrate that this framework can provide some insight to problems in

$$0 = -\partial_x \{a u_\star\} + D \partial_{xx} u_\star - \lambda(x) u_\star + \int_{-\infty}^{\infty} \mu(\Delta) \lambda(x - \Delta) u_\star(x - \Delta) d\Delta. \quad (21)$$

One possible quantity of interest is the first moment of jump locations, so

$$\langle j_\star \rangle = \int_{-\infty}^{\infty} x p_\star dx = \int_{-\infty}^{\infty} x u_\star \lambda dx.$$

Although $\lambda(x)$ is left arbitrary, the Fourier transform can surprisingly be used to gain some insight. Define the Fourier transform of u_\star to be

$$U(k) := \mathcal{F}[u_\star] = \int_{-\infty}^{\infty} e^{-ikx} u_\star(x) dx.$$

Also define the transformed quantities

$$\Lambda(k) := \mathcal{F}[\lambda u_\star], \quad M(k) := \mathcal{F}[\mu]$$

from which, we note that

$$\Lambda'(0) = -i\langle j \rangle, \quad M'(0) = -i\langle \mu \rangle.$$

Taking the transform of (21),

$$0 = -aikU - Dk^2U - \Lambda(k) + M(k)\Lambda(k), \quad (22)$$

which, evaluated at $k = 0$ yields

$$[1 - M(0)] \Lambda(0) = 0. \quad (23)$$

However, we know $\mu(\Delta)$ and $u_\star \lambda = p_\star$ are both probability densities and consequently $M(0) = 1$ and $\Lambda(0) = 1$. Thus, (23) is trivially satisfied. This is not seemingly useful on its own, but we can couple the higher order moments by taking a k derivative of (22) to yield

$$0 = -aikU' - aiU - 2DkU - 2k^2U' - \Lambda' + M'\Lambda + M\Lambda',$$

which, at $k = 0$ and using $M(0) = 1, \Lambda(0) = 1$ yields

$$0 = -aiU - \Lambda'(0) + M'(0) + \Lambda'(0),$$

which says that necessarily

$$M'(0) = -i\langle \mu \rangle = -aiU(0)$$

partnership with Fourier analysis. For this example, take a constant drift and diffusion with random jump size distribution $\mu(\Delta)$. Then, in stationarity, (8) becomes

However, recall from (12) that $U(0) = \langle \tau_\star \rangle$, the mean interjump time, so

$$\langle \tau_\star \rangle = -\frac{\langle \mu \rangle}{a}. \quad (24)$$

This result is quite interesting for two reasons. For one, we have an explicit expression for the mean interjump time revealed by the Fourier transform, independent of the choice of $\lambda(x)$. More interestingly is the sign of the quantity. Since (24) represents an interjump time, it must be non-negative, meaning this quantity only exists if $a, \langle \mu \rangle$ differ in sign. This is intuitive as the drift and jump process must oppose each other to have a chance of reaching stationarity, which this demonstrates.

C. Ornstein-Uhlenbeck with Gaussian jump rate

This example serves to identify curious subtle behaviors of the coupling between diffusion and state-dependent jumps illuminated by our results. Consider, for this example, an Ornstein-Uhlenbeck diffusion with drift $-ax$ and diffusion coefficient D . The jump rate is taken to be Gaussian

$$\lambda(x) = \sqrt{\frac{\alpha^2 \beta}{2\pi}} \exp\{-\beta x^2/2\} \quad (25)$$

and the jump is a fixed distance Δ . This could perhaps be thought of as a molecular motor [30] attached via a Hookean linker, causing a relaxation to an un-stretched $x = 0$ state. As the motor becomes less stretched, it preferentially takes a step of size Δ at a faster rate. Of possible interest would be to study how the cargo diffusion coefficient D effects the effective rate at which the motor steps. The resulting stationary relationship (8) becomes

$$0 = \partial_x \{axu_\star\} + D \partial_{xx} u_\star - \lambda(x) u_\star + \lambda(x - \Delta) u_\star(x - \Delta). \quad (26)$$

There does not appear to be a fruitful approach to studying (26) analytically. However, we solve it numerically using a Crank-Nicolson upwind scheme [31]. From this, u_\star

and consequently p_* is obtained. The first moment of the interjump time $\langle\tau_*\rangle$ is immediate after a quadrature (13), but higher moments require solving differential equation solving, as described by **Appendix A**. Instead of taking this approach, once p_* is obtained, $\mathcal{J}p_*$ is used as a starting condition for the absorbing process 3, which is solved numerically using the same PDE solver. The advantage of this is that the full distribution of τ_* is obtained using the relationship described by **Theorem 1**.

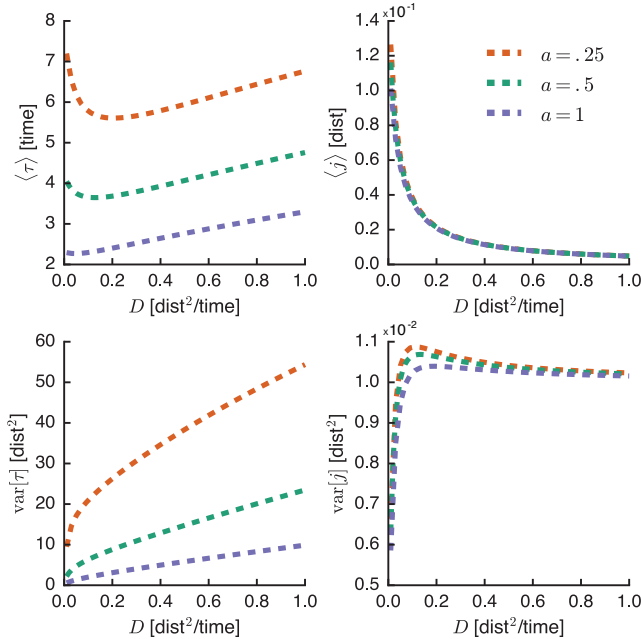


FIG. 3. The first and second moments of the stationary jump locations j_* and interjump times τ_* of the process described by (25),(26) as the strength of diffusion is varied, computed by solving (26) numerically. A non-monotonic dependence on diffusion is seen in the mean interjump time. Parameter values used are $\alpha = 1, \beta = 100$.

The results of this procedure as a function of the diffusion strength D are shown in Figure 3. Plotted are the first and second moments of the stationary jump locations j_* and interjump times τ_* . A noteworthy behavior is observed: for certain parameter values, the mean interjump time is a non-monotonic function of the diffusion strength, D . Although this is perhaps counterintuitive, discrete events with non-monotonic dependence on the strength of diffusion have been observed in other biophysical phenomena [32]. The interjump times seem to change significantly for different values of the drift, the jump locations seem to change relatively little. Intuitively, the variance of both quantities increases with diffusion strength and lower drift.

This example provides an illustration that even when analytical solutions can not be found, the framework provides relationships that can be computed numerically to illuminate new behaviors not identified in previous work.

V. CONCLUSION

In this work, we have presented a mathematical framework for studying jump-diffusion systems with state-dependent jump rates. The formulation is flexible enough to accommodate a variety of behaviors, providing relevance to a wide range of applications. The particular objects of focus in this work are the distribution of the jump locations: the values of X_i at the jump times t_i .

We first reformulate the full process into another that is equivalent between jumps. Using this formulation, the explicit distribution of the jump locations can be extracted and related to the previous jump location. This relation allows us to derive an iterated map description of the sequence of jump locations. With explicit knowledge of the distribution of jump locations, moments of the interjump times can be computed in more generality than previous work [13]. Assuming the process reaches stationarity, the distribution of the full process and the stationary jump locations are shown to be closely related. A brief discussion of possible conditions on convergence to stationarity in the lens of this framework are provided, utilizing tools from theoretical ecology motivated by the observation that the iterated map is effectively a non-negative integral linear operator.

Three examples of particular choices of the diffusion and jump components are briefly discussed. The first shows that models with relatively simple functional forms can be solved exactly using results from our work and likely would not be possible or difficult to derive otherwise. The second example shows that although studying full jump-diffusion models directly using Fourier space may be infeasible, translating the problem to our proposed framework and then studying via the Fourier transform may serve useful. The last example is computed numerically from results derived in the paper and shows that diffusion can have a non-monotonic influence on when coupled with state-dependent jump processes, showing that our results are useful in finding subtle behaviors even when only numerical approaches are feasible.

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Appendix A: Higher order interjump time moments

In theory, higher order moments of the interjump times can be computed with knowledge of the sequence of jump distributions p_i as was discussed with the first moment in (12).

To illustrate this, consider (14) and take the second

moment with respect to τ_i on both sides, resulting in

$$\langle \tau_i^2 \rangle = - \int_{-\infty}^{\infty} \int_0^{\infty} t^2 \partial_t \hat{p}_i dt dx.$$

Again, noting that $u_i = p_i/\lambda = \int_0^{\infty} \hat{p}_i dt$, after integrating by parts, we get

$$\langle \tau_i^2 \rangle = - \int_{-\infty}^{\infty} v_i dx,$$

which looks similar to (12), however, v_i satisfies

$$[\lambda - \mathcal{L}]v_i = u_i. \quad (\text{A1})$$

Thus, computation of the second moment requires solving the differential relationship (A1). Continuing to higher order moments using same approach yields a further hierarchy coupled in a differential manner. Interestingly, the differential operator (left hand side) of (A1) is exactly \mathcal{T} , the same as (6), meaning the Green's function (effectively \mathcal{T}^{-1} , discussed more in **Appendix B**) could perhaps be reused to compute these quantities.

Appendix B: Spectral properties of iterated map

Note that (6) involves the inverse of a differential linear operator \mathcal{T} , which is exactly determined by its corresponding Green's function. Let $G(x, \xi)$ be the corresponding Green's function to \mathcal{T} , meaning that

$$\mathcal{T}_x G(x, \xi) = \delta(x - \xi).$$

Then, (6) can be rewritten as

$$u_{i+1} = \int_{-\infty}^{\infty} G(x, \xi) \mathcal{T} \lambda u_i(\xi) d\xi.$$

After a change of variables, define \tilde{G} by

$$u_{i+1} = \int_{-\infty}^{\infty} \tilde{G}(x, \zeta) u_i(\zeta) d\zeta := \int_{-\infty}^{\infty} G(x, \xi) \mathcal{T} \lambda u_i(\xi) d\xi.$$

For example, for fixed jump sizes: $\mathcal{T}p = p(x - \Delta)$, then

$$\tilde{G}(x, \zeta) = G(x, \zeta + \Delta) \lambda(\zeta + \Delta).$$

Abbreviate this linear non-negative integral operator

$$\mathcal{A}q := \int_{-\infty}^{\infty} \tilde{G}(x, \zeta) q(\zeta) d\zeta.$$

Now, iterations of our map correspond to iterating the integral operator \mathcal{A} with kernel \tilde{G} . This formulation is exactly that of the so-called *integral projection models* (IPM) in theoretical ecology. [33] Stability results from the IPM literature can be used to understand the convergence of our iterative procedure.

The main result comes from [34] and is effectively a statement of the Krein-Rutman theorem [35, 36], the infinite dimensional analog of the Perron-Frobenius theorem. [25, 37]

Although L^1 seems like the natural function space to study the spectral properties of these operators, since they must preserve probability, L^2 turns out to be far more accessible due to issues establishing compactness in L^1 . In Appendix C of [33], the authors provide a more thorough commentary on these complications. In L^2 , we have the property

Theorem B.1 ([36]). *If*

$$\iint_{\mathbb{R}^2} |\tilde{G}(x, \xi)|^2 dx d\xi < \infty,$$

then the integral operator with kernel \tilde{G} maps from L^2 to L^2 .

We then cite the main theorem which establishes the existence of a dominant eigenvalue for \mathcal{A} .

Theorem B.2 (Easterling 1998, [34]). *Suppose that $\tilde{G} \in L^2$ and is non-negative, then if there exists an $\alpha > 0, \beta > 0, u_0$ such that*

$$\alpha(x)u_0(\xi) \leq \tilde{G}(x, \xi) \leq \beta(x)u_0(\xi)$$

for all x, ξ , then the integral operator \mathcal{A} has a dominant eigenvalue with associated eigenfunction.

This establishes conditions for the existence of a dominant eigenvalue and eigenvector, which establish the long-term behavior.

Theorem B.3 (Easterling 1998, [34]). *Assuming that \mathcal{A} satisfies the previous condition, then the stationary distribution u_* is given by the eigenfunction ϕ_1 associated with the dominant eigenvalue λ_1 ,*

$$\lim_{i \rightarrow \infty} \frac{u_i}{\lambda_1^i} = \kappa \phi_1,$$

where κ is a scaling parameter.

The previous theorem suggests that the sequence u_i converges if and only if $|\lambda_1| = 1$. Thus, this summary provides a rough heuristic and novel angle in determining whether the sequence u_i (and the full process) approach stationarity, although this is not the focus of this work.

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